Some applications of uniform boundedness Principle

Corollary:- A non-empty subset S of a normed linear space E is bounded $\Leftrightarrow$ $f(S)$ is a bounded set of numbers for each $f \in E^*$.

A.U. 94 Or, Q.N. ⇒ Show that a non-empty subset of a normed linear space N is bounded if and only if $f[S]$ is bounded set of numbers for each $f \in N^*$.

Proof:- Let S be bounded in E. Therefore there exist a +ve real number α such that $\|x\| \leq \alpha$ for every $x \in S$. If $f \in E^*$.

$\therefore |f(x)| \leq \|f\| \cdot \|x\| \leq \alpha \|f\|$ for every $x \in S$ so $f(S)$ is a bounded set of numbers. Conversely we suppose $f(S)$ is bounded for each $f \in E^*$. For each fixed $x \in S$, we define a linear functional F_x on the dual space E^* by setting $F_x(f) = f(x)$, for each $f \in E^*$, it is easily to verify that F_x is linear. Now,

$$\|F_x\| = \sup_{\|f\|=1} |F_x(f)| = \sup_{\|f\|=1} |f(x)| = \|x\| \quad \text{--- (1)}$$

Hence, F_x is ~~linear~~ a Continuous linear functional on the Banach space E^*. By the hypothesis for each $f \in E^*$,

$$\sup_{x\in S} |F_x(f)| = \sup_{x\in S} |f(x)| < \infty$$

Therefore by (1) the Principle of uniform bounded

$$\sup_{x\in S} \|x\| = \sup_{x\in S} \|F_x\| < \infty.$$

Hence, S is bounded.

NOTE:- पेपर में Or वाला जो Q No. पूछा गया है उसमें E और E* के जगह पर N तथा N* लिखना है

Corollary:- Let E and F be normed linear spaces and $g: E \to F$ be a linear transformation. Then g is Continuous iff $h \circ g \in E^*$ for every $h \in F^*$.

Proof:- A linear transformation $g: E \to F$ is Continuous iff the set $E = \{g(x) \in F : x \in E, \|x\| \le 1\}$ is bounded in F iff $h(E) = \{(h\circ g)(x) : x \in E, \|x\| \le 1\}$ is bounded in K for every $h \in F^*$

Therefore, iff $(h \circ g) \in E^*$, for every $h \in F^*$.

Corollary:- Let E & F be Banach spaces. Suppose $\{T_n\}$ is a sequence of Continuous linear transformation of E into F such that $\{T_n(x)\}$ Converges for each $x \in E$. Then the function T defined by $Tx = \lim_{n\to\infty} T_n x$, $x \in E$, is a Continuous linear transformation of E into F and the sequence $\{\|T_n\|\}$ is bounded.

Proof - Since $\{T_n x\}$ is Convergent for each fixed $x \in E$, It follows that $\{T_n(x)\}$ is bounded for each fixed $x \in E$. Hence by the Principle of uniform boundedness,

$$\sup_n ||T_n|| = m < \infty$$

and $||Tx|| = \lim_{n \to \infty} ||T_n x|| \leq m ||x||$

Hence, $||T|| \leq m$ therefore $\{||T_n||\}$ is bounded.

Q No → The normed linear space X is not Complete.

Proof:- We may write a Polynomial $x(t) \in X$, of degree N_x in the form

$$x(t) = \sum_{j=0}^{\infty} a_j t^j \text{ where } a_j = 0 \text{ for } j > N_x.$$

If $x \in X$, then we Construct a sequence of functionals f_n by the definition.

$$f_n(0) = 0 \ \& \ f_n(x) = a_0 + a_1 + \cdots + a_{n-1} \quad \text{—— (1)}$$

It is a matter of Simple verification that f_n is linear for each n.

Also, $|f_n(x)| \leq n \cdot \max_j |a_j| \leq n ||x||$

So that f_n is bounded. Therefore f_n is a Continuous linear functional for each n.

If $x \in X$ then $x(t)$ is a Polynomial of degree N_x which has at most $N_x + 1$ non zero coefficients and therefore by (1)

$$|f_n(x)| \leq (N_x + 1) \max_j |a_j|$$

for each n where $\max_j |a_j|$ is taken over $a_0, a_1, a_2, \dots a_{N_x}$

Therefore, the sequence $\{f_n(x)\}$ is bounded for every $x \in X$. On the other hand, if

$$x(t) = 1 + t + t^2 + \dots + t^n, \text{ then}$$

$$\|x\| = 1.$$

$$\therefore f_n(x) = 1 + 1 + \dots + 1 = n = n\|x\|.$$

So, $\|f_n\| \geq \dfrac{|f_n(x)|}{\|x\|} = n$. So that $\{\|f_n\|\}$ is not bounded.

Therefore, X is not complete.